REPORT No. 349

A PROOF OF THE THEOREM REGARDING THE DISTRIBUTION OF LIFT OVER THE SPAN FOR MINIMUM INDUCED DRAG

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SUMMARY

The proof of the theorem that the elliptical distribution of lift over the span is that which will give rise to the minimum induced drag has been given in a variety of ways, generally speaking too difficult to be readily followed by the graduate of the average good technical school of the present day. In the form of proof herewith, an effort is made to bring the matter more readily within the grasp of this class of readers. The steps in the proof, briefly outlined, are as follows:

1. Given a basic distribution of lift across the span denoted by (a) with a second supplementary distribution denoted by (1). Then it is shown that the induced drag of lift (a) in the downwash due to lift (1) and the induced drag of lift (1) in the downwash due to lift (a) are equal, and that the total effect of the small distribution (1) on the induced drag will be measured by twice either of these small quantities.

2. Next two small changes are assumed in a basic distribution (a). These are represented by (1) and (2), and are further assumed to be equal in amount and opposite in algebraic sign, thus leaving the original lift unchanged in amount, but changed in distribution. Under these conditions it is then shown that in order for the distribution (a) to be that for minimum induced drag, the change in induced drag due to this small change in distribution must be zero.

3. It is next shown that for any pair of small changes such as (1) and (2) the only value of the basic downwash which will meet the condition of step (2) is downwash constant across the span.

4. It is known mathematically that the elliptical distribution across the span is that which gives a constant value of the downwash and hence as a result of (1), (2), (3), this must be the distribution which will give the minimum value of the induced drag.

The theorem setting forth the elliptical distribution of lift over the span of an airplane as that which will give rise to the minimum induced drag is one of fundamental importance in connection with the study of many problems in aerodynamic theory. Its proof has been given by the use of the calculus of variations and otherwise, but hitherto, so far as the present writer has noted, in form and procedure adapted rather to the expert than to the graduate of the average good technical school of the present day. The present proof has been prepared with the view of bringing a better understanding of this important theorem more readily within the grasp of the technical graduate who, so far as mathematics is concerned, has no more than a fair grasp of the elements of differential and integral calculus.

We at first state the general aerodynamic elements involved in the problem, and with which we assume the reader to be familiar.

NOTATION

Let x denote any point on the span of the plane, reckoned from the center as origin.

$y\,dx$ = lift for a short length of span dx at the point x.

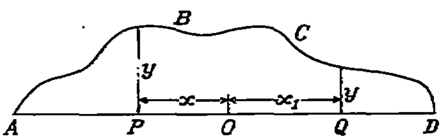

FIGURE 1

Γ = the circulation, or vortex strength at the point x.

w = the induced or downwash velocity at any point determined by an abscissa x_1.

b = the half span.

V = velocity of plane.

ρ = density of medium.

D = induced drag.

Let the distribution of lift across the span be given by any curve as in Figure 1.

We make only two assumptions regarding the character of this distribution.

(1) The value of y is zero at each end.

(2) The curve $ABCD$ has no abrupt changes in curvature.

That is, the value of dy/dx varies continuously along the curve from A to D.

Then the elementary theory of lift gives $y = \Gamma V \rho$. But V and ρ are constant for the plane and therefore y is everywhere proportional to Γ and vice versa.

Consider any point P on the span determined by the abscissa x. Then at this point, and generally,

$$d\Gamma = \frac{d\Gamma}{dx}dx = \frac{1}{V\rho}\frac{dy}{dx}dx$$

is an element of change in Γ and as such is a measure of the strength of the elementary trailing vortex given off at such point.

Now let $y=f(x)$ be the equation to the curve $ABCD$. Then we have

$$\frac{dy}{dx}dx=f'(x)\,dx \text{ and}$$

$$d\Gamma=\frac{1}{V\rho}f'(x)\,dx$$

Then the theory of induced or downwash velocity gives, for the point Q,

$$dw=\frac{1}{4\pi}\frac{d\Gamma}{(x_1-x)}dx=\frac{1}{4\pi V\rho}\frac{f'(x)\,dx}{(x_1-x)}$$

as the measure of the element of w at Q due to the action of the vortex filament at P. (Reference 1.)

Then the entire w at Q will be given by the integration of the above expression across the span or from A to D. This gives:

$$w=\frac{1}{4\pi\ V\rho}\int\frac{f'(x)\,dx}{x_1-x} \qquad (1)$$

Again the theory of induced drag gives, for the element of D at the point Q.

$$dD=y\frac{w}{V}dx_1$$

where y and w both refer to the point Q. Hence:

$$D=\frac{1}{V}\int yw\,dx_1 \qquad (2)$$

Equations (1) and (2) are basic with reference to the theorem which we have now to develop and, as

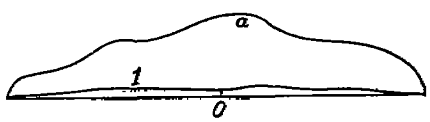

FIGURE 2

noted above, it is assumed that the reader is familiar with the general features of the vortex theory of lift and with the establishment, from fundamental principles, of these two equations.

We shall now proceed to establish the theorem in question by a series of steps which may be listed as follows:

(1) Given a basic distribution of lift across the span. Denote this by (a). Given a second small or supplementary distribution. Denote this by (*1*). Then we have to show that the induced drag of lift (a) in the downwash due to lift (*1*) and the induced drag of lift (*1*) in the downwash due to lift (a) are equal, and that the total effect of the small distribution (*1*) on the induced drag will be measured by twice either of these small equal quantities.

(2) We next suppose two small changes made in a basic distribution (a). These may be represented by (*1*) and (*2*). These are further assumed to be equal in amount and opposite in algebraic sign, thus leaving the original lift unchanged in amount, but changed in distribution. Then under these conditions we have to show that in order for the distribution (a) to be that for minimum induced drag, the *change* in induced drag due to this small change in distribution must be zero.

(3) We have next to show that for any pair of small changes in distribution such as (1) and (2) the only

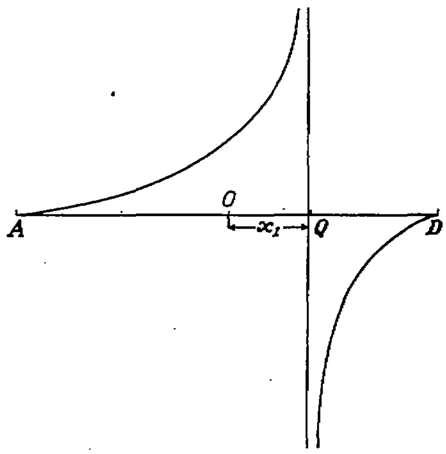

FIGURE 3

value of the basic downwash w_a which will meet the condition of step (2) is w_a=constant, or downwash for the basic distribution=constant across the span.

(4) It is known mathematically that the elliptical distribution of lift across the span is that which gives a constant value of the downwash w and hence is that which will give the minimum value of the induced drag.

We shall now proceed with these steps in order.

(1) In Figure (2) let (a) represent a basic distribution of lift across the span and (*1*) a second or small supplementary distribution. Then in equation (1) we may omit the constant factor and write:

$$w\sim\int\frac{f'(x)\,dx}{(x_1-x)} \qquad (3)$$

Let us integrate this expression in accordance with the formula

$$\int u\,dv=uv-\int v\,du$$

Where u represents $\frac{1}{(x_1-x)}$ and dv denotes $f'(x)\,dx$. We shall then have—

$$w\sim\int_{-b}^{+b}\frac{f'(x)\,dx}{(x_1-x)}=\frac{f(x)}{(x_1-x)}\Big]_{-b}^{+b}-\int_{-b}^{+b}\frac{f(x)\,dx}{(x_1-x)^2} \qquad (4)$$

where $-b$ and $+b$ denote the limits of the integration at A and D, Figure 1.

Taking the first term on the right, special caution must be exercised in making the evaluation due to the fact that in passing from $x=-b$ to $x=+b$, the denominator passes through 0. The range of values of this expression will be therefore as indicated in Figure (3).

At A the value of $f(x)$ is zero, but a zero approached through positive values. We may therefore term such a zero $a+0$ and similarly one approached through negative values, $a-0$. We shall then have at A, $f(x)/(x_1-x)=+0$. Then as the point under consideration approaches Q from A, the value of (x_1-x) will approach a positive zero and the value of $f(x_1)\div(x_1-x)$ at Q becomes $+\infty$. Then as the point passes beyond Q where $x>x_1$, the zero shifts from positive to negative and the value just on the right of Q becomes $-\infty$ and then gradually decreases numerically as x increases, finally approaching 0 on the negative side at D.

The total change in the value of the expression from A to D is therefore from $+0$ to -0 passing through $+\infty$ and $-\infty$ on the way. Suppose, however, that we take two points Q_l and Q_r, respectively on the left and right of Q, and at a very small distance e from Q. Then if for brevity we represent the expression $f(x)/(x_1-x)$ by a single letter φ, we may write—

$$\varphi\Big]_A^D=\varphi\Big]_{Q_l}^{\lim.\, e=0}-\varphi_A+\varphi_D-\varphi\Big]_{Q_r}^{\lim.\, e=0}$$

But as we have seen, φ_A and $\varphi_D=0$ and hence

$$\varphi\Big]_A^D=\varphi\Big]_{Q_l}^{\lim.\, e=0}-\varphi\Big]_{Q_r}^{\lim.\, e=0}$$

But at Q_l, $f(x)=f(x_1-e)$ and $(x_1-x)=e$. Likewise at Q_r $f(x)=f(x_1+e)$ and $(x_1-x)=-e$. Hence we have finally, restoring the original form,

$$\frac{f(x)}{x_1-x}\Big]_A^D=\left[\frac{f(x_1-e)}{e}-\frac{f(x_1+e)}{-e}\right]_{\lim.\, e=0}$$

This we may write[1]—

$$\frac{2\,f(x_1)}{e}\Big]_{\lim.\, e=0}$$

We shall then have for equation (4)

$$w\sim 2\frac{f(x_1)}{e}\Big]_{\lim.\, e=0}-\int_{-b}^{+b}\frac{f(x)\mathrm{d}x}{(x_1-x)^2} \qquad (5)$$

Now turning to equation (2) we may, for our present purpose, omit the constant V and write

$$D\sim\int yw\mathrm{d}x_1 \qquad (6)$$

Note that in this equation both w and y refer to the point Q and the integration is with reference to x_1.

Applying this equation to the case assumed in Figure 2 we note that the total value of D will result from four constituent elements as follows:

(a) That due to lift (a) in downwash (a)
(b) That due to lift (1) in downwash (1)
(c) That due to lift (a) in downwash (1)
(d) That due to lift (1) in downwash (a)

Then considering (a) as basic, the *change* will be the sum of (b), (c), and (d). But if distribution (1) is small in magnitude, item (b), will be small of the second order and negligible in comparison with the other values. We shall have, therefore, for the resultant effect of the supplementary distribution (1), simply the sum of (c) and (d).

[1] For a somewhat more formal mathematical treatment of this evaluation, see Appendix 1.

Now referring to Figure 2, let us denote by subscripts a and 1 the various quantities referring to these two distributions of lift. Then remembering, as above noted, that in (6) y refers to the point O, as fixed by the abscissa x_1, we may write:

$$y_a=f_a\ (x_1)$$

$$y_1=f_1\ (x_1)$$

and from (5)

$$w_a\sim\frac{2f_a(x_1)}{e}\Big]_{\lim.\, e=0}-\int_{-b}^{+b}\frac{f_a(x)\mathrm{d}x}{(x_1-x)^2} \qquad (7)$$

Next we apply (6) taking w_a and y_1. This will give the induced drag for lift distribution (1) in the downwash caused by distribution (a). We thus have

$$D_{1a}\sim 2\int_{-b}^{+b}f_1(x_1)\frac{f_a(x_1)}{e}\Big]_{\lim.\, e=0}\mathrm{d}x_1\ -\int_{-b}^{+b}f_1(x_1)\int_{-b}^{+b}\frac{f_a(x)\mathrm{d}x}{(x_1-x)^2}\mathrm{d}x_1$$

Or writing the last term as a double integral,

$$D_{1a}\sim 2\int_{-b}^{+b}f_1(x_1)\frac{f_a(x_1)}{e}\Big]_{\lim.\, e=0}\mathrm{d}x_1\ -\int_{-b}^{+b}\int_{-b}^{+b}\frac{f_1(x_1)f_a(x)}{(x_1-x)^2}\mathrm{d}x_1\mathrm{d}x \qquad (8)$$

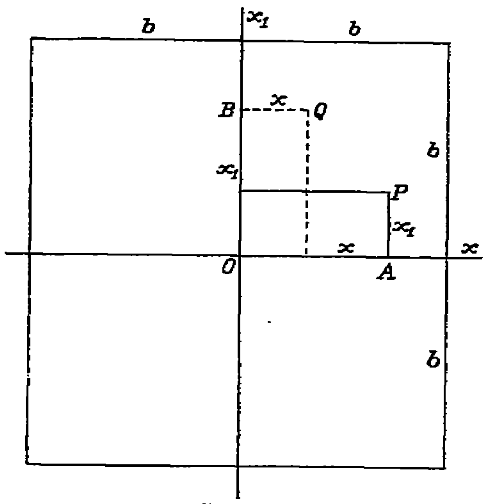

FIGURE 4

Then vice versa if we take y_a and w_1 we shall have the induced drag for lift distribution (a) in the downwash caused by distribution (1). This will give:

$$D_{a1}\sim 2\int_{-b}^{+b}f_a(x_1)\frac{f_1(x_1)}{e}\Big]_{\lim.\, e=0}\mathrm{d}x_1\ -\int_{-b}^{+b}\int_{-b}^{+b}\frac{f_a(x_1)f_1(x)}{(x_1-x)^2}\mathrm{d}x_1\mathrm{d}x \qquad (9)$$

Now in equations (8) and (9) it is clear that the first terms on the right, even though they do involve infinities, will be equal. For the second terms, we refer to Figure 4. The square with center at 0 and axes of x and x_1 represents the combination of values of x and x_1 to be covered by the integration. That is, every possible combination of values of x and x_1 between $+b$ and $-b$ is represented by a point on this square and

the double integral means that for each such point or combination, the value of the expression under the sign of integration is to be found, and these elements summed. Now at a point P we shall have, in equation (8) $f_a(x_1)=f_a(AP)$ and $f_1(x)=f_1(OA)$, while at the point Q we shall have, in equation (9), $f_1(x_1)=f_1(OB)$ and $f_a(x)=f_a(BQ)$. But for any point P there will always be another point Q for which $AP=BQ$ and $OA=OB$. And what is true in the first quadrant will be equally true in the others. Hence for every point representing a value in the numerator of equation (8) there will be another point representing a value in equation (9) such that the two values are equal, each to each. The denominators likewise will be equal since (x_1-x) will have the same numerical value for each point, and in the square the value will be the same regardless of the sign. Hence the summations will be equal and the values of (8) and (9) are equal throughout and as a result, we have, $D_{a1}=D_{1a}$ or—

The induced drag due to y_1 under downwash w_a will equal that due to y_a under downwash w_1. And furthermore since, as we have seen, the total effect is measured by the sum of these two items, it will obviously be measured by twice either one taken singly.

It may be noted at this point that the proof that $D_{1a}=D_{a1}$ is independent of the relative size of distribu-

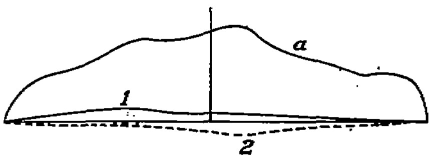

FIGURE 5

tions (a) and (*1*) and that this relation is true generally. For our present purposes, however, we are concerned rather with the particular case where (*1*) is small and where the induced drag due to y_1 in the field w_1 is negligible and ∴ where the total effect is measured as above noted.

(2) The second stage in the proof calls for a consideration of the results of making two small changes in a basic distribution of lift, one positive and one negative, equal in numerical value and of any character of distribution: See Figure 5. Here (*1*) and (*2*) denote any two small distributions of lift, No. (*1*) positive and No. (*2*) negative, equal in numerical value and distributed each in any manner along the span.

Due to the fact that (*1*) and (*2*) are both small it will result that the elements of induced drag due to y_1w_1, y_1w_2, y_2w_1 and y_2w_2 will all be small of the second order and therefore negligible in comparison with the other values. It will result then that the total *change* in induced drag will be measured by the components:

D_{a1} = drag due to y_a in field of w_1
D_{1a} = drag due to y_1 in field of w_a
D_{a2} = drag due to y_a in field of w_2
D_{2a} = drag due to y_2 in field of w_a

But as we have seen,

$$D_{a1}=D_{1a} \text{ and } D_{a2}=D_{2a} \text{ and hence}$$
$$D_{a1}+D_{1a}+D_{a2}+D_{2a}=2(D_{1a}+D_{2a})$$

That is, the total *change* in the induced drag as a result of these changes in the distribution of lift (the total lift remaining the same) will be twice the sum of the induced drags due to y_1 and y_2, both in the downwash w_a.

Now this total change may be (−), (+), or 0. If it is (−), it will show that the original distribution y_a was one which admitted of reduction in the induced drag for the same total lift, and hence that it was not the distribution for minimum induced drag.

If it is (+), it would be possible, by interchanging the signs of the distributions (*1*) and (*2*), to make it (−); that is, by making distribution (*1*) negative and (*2*) positive, we should interchange the numerical relations and the result would change sign. This again would show that the original distribution y_a is not that for the minimum induced drag.

If then the value $2(D_{1a}+D_{2a})$ is anything other than zero, it will indicate that the basic distribution y_a is one which will give an induced drag permitting of reduction (by redistribution) by an amount at least measured by this value. Hence the only distribution of y_a which will give an induced drag not permitting of reduction is that which will give zero for the value of the change $2(D_{1a}+D_{2a})$. But the condition for this is $D_{1a}=-D_{2a}$. From equation (6) this will mean:

$$\int y_1w_a\,dx_1=-\int y_2w_a\,dx_1 \tag{10}$$

while at the same time we have the condition:

$$\int y_1\,dx_1=-\int y_2\,dx_1 \tag{11}$$

(3) To carry out step (3) of the proof, we note that the simultaneous fulfillment of equations (10) and (11) may be realized in two ways.

(1) For any and all pairs of distributions y_1 and y_2 which meet equation (11) the only value of the basic downwash w_a which will meet equation (10) is w_a = constant.

(2) For any particular pair of distributions y_1 and y_2 fulfilling equation (11), distributions of w_a may be found, not necessarily uniform in value, meeting equation (10) for this special pair y_1, y_2 but failing for other pairs which still meet equation (11).

In order not to interrupt the main argument at this point, further reference to this point is transferred to an appendix.

It appears, however, that the fulfillment of equation (10) for any and all possible pairs of distributions y_1 and y_2 which meet equation (11) (as is implied by the character of the preceding argument) can only be realized if w_a is a constant multiplier under the sign of integration.

That is, in order that the original distribution y_a shall be that for minimum induced drag, it must be such a distribution as will give a constant value of the downwash w across the span. Under these conditions then, a small supplementary induced drag $\frac{1}{V}\int y_1 w_a dx_1$, will equal numerically a second small supplementary induced drag $\frac{1}{V}\int y_2 w_a dx_1$, provided $\int y_1 dx_1 = \int y_2 dx_1$; and as a result, where y_1 and y_2 are opposite in sign, we shall have $D_{1a} = -D_{2a}$ or $D_{1a} + D_{2a} = 0$, or no change in the over-all induced drag for this small change in distribution. This condition, as we have seen, implies that the basic induced drag is at its minimum value for the total lift as given by the distribution y_a.

(4) Now it is known that the elliptical distribution across the span is that which will give a uniform value of the downwash velocity w. That is, it is known that for the elliptical distribution the value of $\int_{-b}^{+b}\frac{f'(x)dx}{x_1-x}$ is a constant and equal to π. It is no part of the present paper to prove this particular result. It is a matter of integral calculus, pure and simple, and while the integration requires special care due to the fact that in passing from $-b$ to $+b$ the denominator becomes 0 and infinite values are involved, it is here assumed that this particular result is either known or accepted. (Reference 2.)

The proof is thus complete that the ellipitcal distribution of lift across the span is that which will give the minimum value of the induced drag for the given total lift so distributed.

An interesting side light on this problem may be obtained by considering the results of a continued

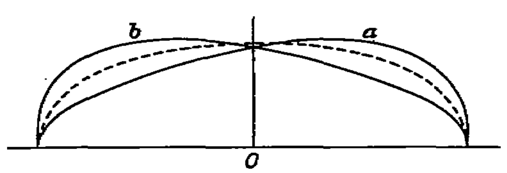

FIGURE 6

series of small changes in distribution (the total remaining the same) such as to gradually change the total distribution from something like that of (a) over into (b) (See Fig. 6) and passing through the elliptical form at a mid-point as indicated by the dotted line. Such a series of changes could, of course, be realized by the continued application of a series of equal plus and minus small changes. In Figure 7 let x x denote

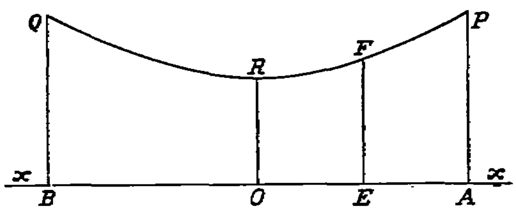

FIGURE 7

an axis of change and let A be the point for distribution a, B the point for distribution b and 0 that for the elliptical distribution. Then AP may represent the value of the induced drag for distribution a, BQ for distribution b and OR for the semi-ellipse. Then the entire series of changes by which a is transformed into b will give a series of values of the induced drag as indicated by the curve PRQ and it is seen how any small change from a distribution represented at E will result in a slight decrease or increase in the value of the ordinate EF depending on whether the change is toward or away from the elliptical distribution. This will indicate, as we have seen, that EF is not a minimum ordinate and that the distribution of lift indicated at E is not that for minimum induced drag. On the other hand, with the distribution at 0 (elliptical) and the value of the induced drag OR, any small change either way will make no sensible change in the ordinate OR; or otherwise the sum total of the change in OR will be zero and this will indicate that OR is a minimum ordinate and that the distribution at 0 (elliptical) is that for the minimum value of the induced drag.

REFERENCES

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2. ———: Aviation, October 19, 1929, page 826.

APPENDIX I

For the evaluation of the expression $f(x)/(x_1-x)$ between the limits of $-b$ and $+b$, we note first that the function is assumed to be continuous and without singularities. At any point near Q, distant e, we may therefore develop $f(x)=f(x_1\pm e)$ by Taylor's theorem.

Thus for the point Q_r on the right of Q we shall have:

$$f(x)=f(x_1)+f'(x_1)e+\frac{f''(x_1)}{2}e^2+\text{etc.}$$

And for the point Q_l on the left of Q we shall have:

$$f(x)=f(x_1)-f'(x_1)e+\frac{f''(x_1)}{2}e^2-\text{etc.}$$

Hence denoting by φ the function to be evaluated, we shall have:

$$\varphi\Big]_{Qr}=\frac{f(x_1)}{x_1-x}+\frac{f'(x_1)e}{x_1-x}+\frac{f''(x_1)}{2(x_1-x)}e^2$$

$$\varphi\Big]_{Ql}=\frac{f(x_1)}{x_1-x}-\frac{f'(x_1)e}{x_1-x}+\frac{f''(x_1)}{2(x_1-x)}e^2$$

But for Q_r, $(x_1-x)=-e$ and for Q_l, $(x_1-x)=+e$. Hence we shall have:

$$\varphi\Big]_{Qr}=-\frac{f(x_1)}{e}-f'(x_1)-\frac{f''(x_1)}{2}e$$

$$\varphi\Big]_{Ql}=\frac{f(x_1)}{e}-f'(x_1)+\frac{f''(x_1)}{2}e$$

Toward the limit the terms having e in the numerator with powers of 1 and higher will vanish and we shall have:

$$\varphi\Big]_{Ql}-\varphi\Big]_{Qr}\Big]_{\lim.\ e=0}=\frac{2f(x_1)}{e}\Big]_{\lim.\ e=0}$$

APPENDIX II

Conditions under which we may have—

$$\int y_1 dx = \int y_2 dx \text{ and} \tag{1}$$

$$\int y_1 w dx = \int y_2 w dx \tag{2}$$

In Figure 8 let y_1 and y_2 fulfill equation (1). Then such condition may be expressed by

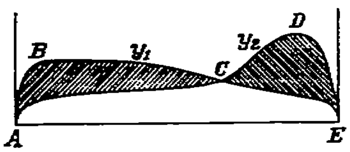

FIGURE 8

Area ABC = Area CDE

Now it is obvious that as we multiply the ordinates of y_1 and y_2 by w as a factor, we shall likewise multiply the ordinates of these *difference* areas by the same factor w. Then in order to insure the fulfillment of equation (2), we have only to insure the multiplication of these areas ABC and CDE, as a whole, by the same factor. Thus if we multiply the areas ABC and CDE both by some number m, the enlarged areas will still be equal numerically, and equation (2) will be fulfilled. Now we can multiply these areas by m by multiplying each and every ordinate by m, or otherwise by various other arrangements of the factor w, such however that, as a whole, the areas ABC and CDE will both be multiplied by the same over-all factor m. These special distributions of w all depend on the particular pair of distributions y_1, y_2, and would not hold in general for other distributions which still fulfill equation (1).

On the other hand if w is a constant factor, then no matter what the pair of distributions y_1 y_2, equation 2) will be fulfilled. In such case we can, of course, take the w out from the sign of integration and write for equation (2)

$$w\int y_1 dx = w\int y_2 dx$$

which is the same as (1)

But it is clear from the manner in which these small supplementary distributions y_1 and y_2 are used in developing the line of proof, that they are entirely unrestricted as to form and character, so long as equation (1) is fulfilled. Or otherwise, to any one basic distribution giving a downwash denoted in general by w, we may, without in any way affecting the argument, apply any and all types and forms of small supplementary distribution y_1 and y_2, so long as equation (1) is fulfilled, and the only single distribution of w which will satisfy equation (2) with *any* and *all* of these possible pairs y_1 and y_2, is w = constant. And thus this link in the general proof is established.

APPENDIX III

For those who may be interested, the following brief outline of the formula, Lift $= \Gamma\rho V$ is given. The details may be found in any good text on fluid mechanics.

Given a circular cylinder of infinite or indefinite length placed in an infinite stream of fluid having two component motions:

(1) A horizontal motion with velocity V in a direction $\perp$ to the axis of the cylinder. Assume this to be from right to left.

(2) A motion of rotation (assumed left handed) about the axis of the cylinder with velocity at any distance r measured by $\Gamma/2\pi r$.

Then for any circle with radius r, the product of the length of the circumference by the velocity will be the constant Γ. This is called the circulation, or vortex strength per unit length of the cylinder.

For an element of length dx, the corresponding vortex strength will be Γdx.

Let ρ be the density of the fluid. Then for the total force reaction on the cylinder per unit length, the principles of the mechanics of fluids (see any textbook) give:

$$F = \Gamma\rho V$$

The theorem shows also that this force is directed $\perp$ to the direction of flow of velocity V and such that the body is urged toward the side on which the velocity due to the circulation, or vortex motion, tends to augment the direct velocity V. With the above assumptions as to directions, this will be vertically upward. Such a force is called a "lift."

We next assume a geometrical or imaginary cylindrical surface of radius R surrounding the given cylinder of radius r. The total reaction between the fluid outside this surface and that inside will consist of two parts:

(1) A net resultant pressure acting over this surface and directed the same as the net force on the cylinder itself.

(2) A change of momentum directed downward experienced by the fluid during its passage through the volume lying between the two cylindrical surfaces.

But the time rate of the change of momentum is the measure of a force and the reaction of this change of momentum downward will be a force upward to be considered as acting over the surface of the cylinder of radius R.

The same principles of mechanics will then show that the sum of these two force reactions is again $\Gamma\rho V$. The two parts have values as follows:

$$F_p = \frac{\Gamma\rho V}{2}\left(1 + \frac{r^2}{R^2}\right) \qquad (1)$$

$$F_m = \frac{\Gamma\rho V}{2}\left(1 - \frac{r^2}{R^2}\right) \qquad (2)$$

where F_p denotes the part due to pressure direct and F_m that due to change of momentum.

If $R = r$, $F_m = 0$ and the entire reaction is direct pressure as we have seen. If R is very great, the two parts approach equality at the half value $\Gamma\rho V/2$ and at ∞ these values are reached. The sum, however, as shown by the values in (1) and (2) is always $\Gamma\rho V$.

Now, suppose R very large in comparision with r. Then imagine the cylinder with radius r to be quickly withdrawn and an indefinitely long body with airfoil section substituted in its place, the latter being of the same general order of size as the cylinder of radius r. Or otherwise, consider a separate case, with the airfoil section instead of the cylinder. With R very large compared with the dimensions of the body, it can make no sensible difference in the character of the flow at and through the surface of the outer cylinder. Or otherwise, we may say that we can certainly go to a radius R so great that the flow at and through the surface of a cylinder of this radius will differ insensibly in the one case as compared with the other.

But over this surface of radius R, the total reaction (pressure + momentum change) will be $\Gamma\rho V$.

Consider again the condition of the fluid between the two cylindrical surfaces. It is constantly undergoing a change of momentum as indicated in equation (2). Furthermore, while the force reaction of the fluid which is thus undergoing a change of momentum is upward, or in the same direction as the force F_p, the direction of the change of momentum itself is downward, or opposite to the direction of F_p.

Now this fluid can be acted on from the outside, only by way of pressure transmitted to it over two surfaces—, the surface of the body of airfoil section and the surface of the cylinder of radius R. The final downward momentum must then be the result of the joint action of these two pressures. Call the total force reaction, body to fluid, F. Since the pressure F_p is up, the force F must be down and we shall have

$$F - F_p = F_m \text{ or } F = F_p + F_m = \Gamma\rho V$$

This gives the total force, body to fluid, down and hence the equal force, fluid to body, is upward, and measured by $\Gamma\rho V$, the same value as for the body with circular cross section. Hence regardless of the form of cross section of the body, the total force reaction under the conditions assumed will be a lift measured by $\Gamma\rho V$.

APPENDIX IV

The effect of downwash is to produce a vertically downward current of velocity w about the airplane. compounding with the relative horizontal velocity V. The result is a direction of resultant flow relative to the plane, tipped down at an angle, whose tangent is w/V. Denote this angle by γ. But the lift as defined and as given by formulas and by measurement is always at right angles to the direction of relative wind. Hence with downwash, the direction of the lift L will be tipped back from the vertical at the same angle γ. But lift in this direction will have a backward component $L \sin \gamma$. The angle γ is always small, however, and we may, therefore, without sensible error, take the tangent for the sine and write $L \sin \gamma = Lw/V$. This force opposing the motion of the plane, is called the induced drag. For a small element dx, of the span, it will have the value

$$dD = y\frac{w}{V}dx_1 \text{ as in equation (2)}$$

STANFORD UNIVERSITY,
October 12, 1929.